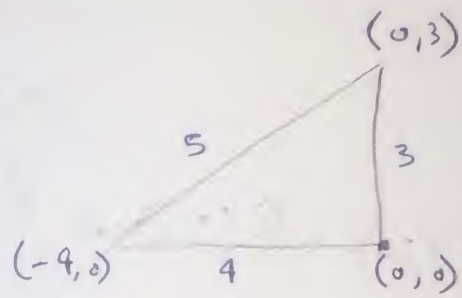
سكشن رياضة رقم (٨)

$$\int_\gamma |f(z)|\, dz \leq ML$$ → طول المنحنى

*show that: $\int_\gamma |e^z - \bar{z}|\, dz \leq 60$ ، when $\gamma$ denote that boundary of tringular $z=0$, $z=-4$, $z=3i$

sol




$L = 3 + 4 + 5$

$\boxed{L = 12}$

$$|f(z)| = |e^z - \bar{z}| \leq |e^z| + |\bar{z}|$$

$$\leq |e^{x+iy}| + |x - iy|$$

$$\leq |e^x \cdot e^{iy}| + \sqrt{x^2+y^2}$$

$$|e^z - \bar{z}| \leq e^x |e^{iy}| + \sqrt{x^2+y^2}$$

$$\leq e^x |\cos(y) + i\sin(y)| + \sqrt{x^2+y^2}$$

$$\leq e^x * \sqrt{\cos^2 y + \sin^2 y} + \sqrt{x^2+y^2}$$

$$\leq e^x + \sqrt{x^2+y^2}$$

$$\text{at } (0,0) \longrightarrow |e^{z} - \bar{z}| \leq 1$$

$$\text{at } (0,3) \longrightarrow |e^{z} - \bar{z}| \leq 4$$

$$\text{at } (-4,0) \longrightarrow |e^{z} - \bar{z}| \leq 4.02 \longrightarrow M$$

$$\int_{\gamma} |f(z)|\, dz \leq 60 \quad \#$$

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## مفكوك تايلور وماكلورين

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots$$

### Important

$\boxed{1}$ $$e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$\boxed{2}$ $$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \cdots$$

$\boxed{3}$ $$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \cdots$$

$\boxed{4}$ $$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} \cdots$$

$\boxed{5}$ $$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} \cdots$$

$\boxed{2}$ Sec 8

$\boxed{6}$ $$\frac{1}{1-z} = 1 + z + z^2 \cdots\cdots$$

$\boxed{7}$ $$\frac{1}{1+z} = 1 - z + z^2 - z^3 \cdots\cdots$$

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* Find Maclurine series:-

$\boxed{1}$ $$f(z) = z\, e^{z-1}$$

$$= z\left[1 + (z-1) + \frac{(z-1)^2}{3!} + \cdots\cdots\right]$$

$\boxed{2}$ $$f(z) = \ln\left(\frac{1+z}{1-z}\right)$$

$$= \ln(1+z) - \ln(1-z)$$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 \cdots$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 \cdots$$

$$\ln(1-z) = \int \frac{1}{1-z}, \quad \ln(1+z) = \int \frac{1}{1+z}$$

$\boxed{3}$ sec 8

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} \cdots\cdots$$

$$\ln(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} \cdots\cdots$$

$$\ln(1+z) - \ln(1-z) = -2\left[\frac{z^2}{2} + \frac{z^4}{4} \cdots\cdots\right]$$

where: $f(z) = \ln(1+z) - \ln(1-z)$

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## "Laurents series"

$\frac{1}{z - z_0}$ ??

* find Laurents series

- $z <$ رقم — نأخذ الرقم عامل مشترك
- $z >$ رقم — نأخذ $z$ عامل مشترك
- $|z-a| <$ رقم — $\boxed{u = z - a}$

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* مثال هام جداً *

$$* \; f(z) = \frac{1}{z^2 - 3z + 2} \qquad 1 < z < 2$$

$$= \frac{1}{(z-2)(z-1)} = \frac{A}{z-2} + \frac{B}{z-1}$$

بمجرد النظر :

$$B = -1 \quad , \quad A = 1$$

$$\therefore f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{-1}{2}\left[\frac{1}{1-\frac{z}{2}}\right] + \frac{-1}{z}\left[\frac{1}{1-\frac{1}{z}}\right]$$

$$= \frac{-1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right] - \frac{1}{z}\left[1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 \cdots\right]$$

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**Singularity**

- removable
- Pole
- essentially

① removable

ـــه قيم أصفار المقام التي عندها البسط = 0

$$\lim_{z \to z_0} \neq \infty$$

$\boxed{ex}$ $\displaystyle\int \frac{\sin z}{z}$

$$z = 0 \Rightarrow \lim_{z \to 0} \frac{\sin z}{z} = 1 \neq \infty$$

$$\Rightarrow \operatorname{Res} f(z) = 0 \qquad \Rightarrow \oint \cdot = 0$$

## 2 Pole

أصفار المقام عدد محدود

### a simple pole

$$\operatorname{Res} f(z) = \lim_{z \to z_0} (z - z_0) f(z)$$

### b Pole of order $(n+1)$

$$\operatorname{Res} f(z)\, dz = 2\pi i \operatorname{Res} f(z)$$

$$= \frac{1}{n!} \lim_{z \to z_0} \frac{d^n}{dz_0^n} (z - z_0)^{n+1} f(z)$$

## 3 essentially

* أصفار المقام عدد لانهائي

$\longrightarrow e^{\frac{1}{z}}$, $\cos \frac{1}{z}$, $\sin \frac{1}{z}$, $\frac{1}{1 - \frac{1}{z}}$, ....

$$\operatorname{Res} f(z) = a_{-1}$$

essentially, Pole ل القانون التالى $\longleftarrow$

$$\boxed{\oint f(z)\, dz = 2\pi i\, f(z)}$$

$\boxed{6}$ Sec 8

$$\operatorname{Res} f(z)\, dz = \frac{1}{n!} \lim_{z \to z_0} \frac{d^n}{dz_0^n} (z - z_0)^{n+1} * f(z)$$

* Find Residue of $f(z) = \dfrac{2z+1}{(z+4)(z-1)^2}$

a) at $z = -4$

$$\operatorname{Res} f(z) = \lim_{z \to -4} (z+4) \frac{2z+1}{(z+4)(z-1)^2}$$

$$= \lim_{z \to -4} \frac{2z+1}{(z-1)^2} = \frac{-7}{25}$$

b) at $z = 1$

$$\operatorname{Res} f(z) = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz} (z-1)^2 * \frac{2z+1}{(z+4)(z-1)^2}$$

$$= \lim_{z \to 1} \frac{2(z+4) - (2z+1)}{(z+4)^2} = \frac{7}{25}$$

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* $\oint \sinh \frac{1}{z} \, dz$

Sol

$$f(z) = \sinh\left(\frac{1}{z}\right) = \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \cdots$$

$$= 1z^{-1} + \cdots$$

$$\operatorname{Res}(f(z)) = 1$$

$\boxed{7}$ Sec 8